Blending and Other Conceptual Operations in the Interpretation of Mathematical Proofs

Adrian Robert*

1 Introduction

Insights[1] from cognitive semantics suggest that conceptual mapping and combination processes play a role at several levels in natural language interpretation. It has been proposed by Lakoff (1987) and Johnson (1987) that such correspondence-making plays a role not only in communicative but in reasoning processes, and, in particular, mathematical reasoning. This proposal suggests that abstract mathematical concepts are grounded in experientially-based image schemas, and reasoning with the concepts proceeds according to the correspondences we make between the properties of the grounding schemas and the specific configurations arising in mathematical situations.

*Department of Cognitive Science; University of California, San Diego; La Jolla, CA 92093-0515; arobert@cogsci.ucsd.edu.

[1]In formulating the ideas contained in this paper I benefited from discussions with Gilles Fauconnier.

The present paper carries this proposal forward by examining in detail how the concepts specified in mathematical definitions are grounded and how they are manipulated in mathematical proofs. As will be shown, proofs are generally not sequences of precise propositions that follow logically from one another as the folk psychological conception would have it, but sets of instructions directing the interpreter to form correspondences between conceptual domains. It is the pattern of mentally observed consonance and/or dissonance arising from the resulting transfer of structural information that leads the interpreter to be convinced of the validity of a proof's conclusion.

The theory of *blended spaces* developed by Fauconnier & Turner (1994) provides a useful framework for studying this conceptual correspondence-making in mathematics and relating it to similar processes in language. This framework has been applied to analyze phenomena at several levels in language ranging from grammar to discourse, as well as certain types of literary analogies and mathematical conceptual developments. Since it is described elsewhere in this volume in the paper by Fauconnier & Turner, we do not explain it further here.

2 The Grounding of Mathematical Concepts

2.1 The Distinction Between Formal and Informal Mathematics

Before discussing mathematical grounding it is necessary to draw a distinction between *formal* and *informal* mathematics (see Lakatos, 1976, pp. 1–5). *Formal mathematics* as a program was originated by David Hilbert (1899/1959)[2] in response to pressures which will be discussed in Section 4.2 and was most thoroughly developed by Whitehead & Russell (1910/1925). It is done within a *formal system* made up of a collection of symbols, a set of axioms or strings of the symbols, and a set of rules specifying how new strings may be derived from old. A *theorem* is a string that has been derived through a sequence of rule applications starting from one of the axioms, and its *proof* is simply the sequence of strings produced by the applications. A theorem can provide the starting point (in place of an axiom) for new proofs. To verify a proof, it is merely necessary to check that the preconditions for the given rule's application are satisfied at each step, and original theorems can be produced simply by the selecting applicable rules and applying them in sequence starting with axiom or theorem.

[2]It has its earliest roots, however, in Euclid's axiomatic approach in the *Elements*, and Peano's (1889) development of symbolic shorthand for proof notation was an important precursor (see discussion in Bell (1945) on pp. 333–4, 558–9).

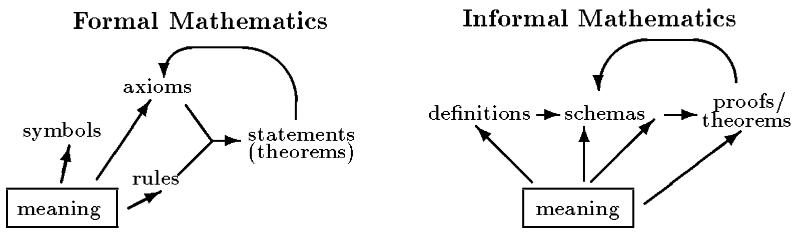

FIGURE 1 Formal and Informal Mathematics

Notice that the production of proofs and theorems is essentially internal to the system: valid chains of applications depend only on the form of the rules and axioms, not on what the symbols stand for. As formal mathematics is typically employed, the symbols *do* stand for external entities or concepts; this is the route by which formal mathematics may be applied to understand and predict phenomena in the experiential world. But the properties of these experiential concepts play a role only in initially determining the forms of the axioms and rules, and subsequently no reference is made to these properties in determining whether a given proof is valid. This latter task is easily automated, and in fact all present machine-based theorem proving systems (see review by Bundy (1983) and Farmer et al. (1995) for a recent example) operate within the formalist program.

This stands in stark contrast to *informal mathematics*, the only form of mathematics extant prior to Hilbert and still the predominant form today. It is the familiar pure mathematics studied in high school and university courses, in which concepts are defined making reference via natural language to schemas and intuitions deriving from everyday experience and then reasoned with in proofs according to their resulting properties. A theorem is a (possibly complex) statement of relationship between defined concepts, and a proof is a sequence of statements of relationship between defined concepts. The validity of each step in a proof is verified by reference to the properties of the concepts. Thus, there is no two-stage separation of *meaning* (by which is simply meant connection to everyday experience) from *mathematical manipulation* as in the case of formal mathematics. The differences[3] are schematically illustrated in figure 1.

[3]In fact there is something of the informal meaning-concept connection that does play a role in formal mathematics, which is that *the ways in which strings of symbols and rule conditions interact* are grounded in experience in the same way that informal mathematical schemas are. The difference is that in formal mathematics this is not the level of meaning of concern to the mathematician. See discussion in Section 4.2 below.

The predominant folk psychological model of mathematical activity corresponds more or less with formal mathematics as it has been described, however in the remainder of this paper we focus on informal mathematics – except in Section 4.2 where the formal and informal versions are compared with one another.

2.2 The Grounding of Informal Mathematical Concepts

In mathematical discourse (textbooks, lectures, etc.), concepts are specified by natural language definitions which make reference to natural concepts or schemas (i.e., those available as a result of commonplace experience) but modify them in one or both of two ways, referred to as *truncation* and *extension*. Truncation is aptly illustrated by some of Euclid's original definitions in the *Elements* (trans. Heath, 1956, p. 153):

1. A **point** is that which has no part.
2. A **line** is breadthless length.
5. A **surface** is that which has length and breadth only.
8. A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie on a straight line.

What is crucial in definitions 1,2,5 is that there is nothing in the world we know that corresponds to the objects they describe. In the real world, all examples of lines have width and even depth, all examples of points have some extension, and all examples of surfaces have depth. Yet the *concepts* of location, length, and breadth *do* exist independently in the mind as useful generalizations for organizing experience. Euclid's definitions, therefore, invite us to consider a very strange yet imaginable kind of objects – those which possess only certain subsets of the attributes which objects possess in the real world.

The subsequent text in the *Elements* explores the relationships between hypothetical instances of the defined objects. We are able to comprehend and understand these relationships because the concepts like length and location that exist in our minds possess a rich network of relational connections with other concepts. We know without Euclid having to tell us, for instance, that points have locations, and lines have locations and orientations. Thus, when definition 8 is given, we understand exactly what is being referred to, for we well know that two directed lengths, when they meet, will do so with a certain quantitative relationship between their directions, and when Euclid later asks us to consider a means for constructing an equilateral triangle from a line segment using compass and straight-edge, we are able to easily see that it works as he describes.

Truncation alone, however, does not fully explain the grounding of Euclidian geometry, let alone other branches of mathematics. The notion of *extension*, based on the idea that we trust certain procedures to result in certain situations, is also necessary. The notion of CONGRUENCE, for example, is based on the imagined possibility of placing figures over one another and noting their correspondences. Based on our own experiences in abstractly similar situations, we understand that such operations could conceivably be performed, even though they never will. Similarly, the notion of NUMBER is based on the idea that certain regular correspondences are maintained in imaginary augmenting and reducing operations on collections of abstract entities just as they are in our experiences with real objects. This kind of grounding is discussed by the philosopher Philip Kitcher (1984) as based on a notion of an "ideal agent" operating in "supertime", and is described by Lakoff (this volume) as involving metaphorical mapping from our schemas for agentive manipulation onto the relevant abstract symbols and structures.

Thus, the properties of mathematical concepts derive from the properties of already-possessed perceptual and motor schemas grounded in experience. Although the schemas are combined and modified in novel ways, the interactive characteristics which, as will be seen below, play the crucial role in mathematical reasoning, are all based on these grounded intuitions. We suggest that the mathematical concepts themselves achieve some independent cognitive reality of their own through processes of entrenchment.

3 Blending in the Interpretation of Mathematical Proofs

In this section, the argument will be made that the interpretation and verification of mathematical proofs involves mapping the general schematic components of defined concepts onto the specific elements of an imagined configurational situation. As mentioned in the introduction, this process is similar to what is proposed to occur in metaphor in language, in which a general schema is applied to understand relations between elements in a particular content domain. We begin by examining the nature of proof in informal mathematics.

3.1 The Nature of Proof in Informal Mathematics

A mathematical proof is a sequence of statements which ultimately lead the reader to perceive a valid connection or relationship between a theorem's conditions and its consequences. This is accomplished by breaking down the relationship, which may be quite complex, into simpler relationships and guiding the reader in such a way that he may convince himself that each of them is true. This process of guided observation has been likened by the mathematician G.H. Hardy (1928) to pointing out the features of a distant mountain range for observation, except that the observed objects are internally imagined instantiations viewed by the mind's eye.

The instantiations are constructed by reference to the mathematical schemas discussed in Section 2, directed by what are termed *introducing* statements. These typically take the form "**Let** x be a y", as in "**Let** E be a set" or "**Let** x be an element of E". The interpretation of such statements involves mapping a label or an imagined entity onto the slot of a given schema; the resulting "meaning" is an *imagined instantiation* of the schematic slot, an object held in mind with certain properties.

Inference statements state a relationship between schemas and instantiations held in mind which it is up to the interpreter to verify for himself. The interpretation of these statements generally involves constructing a mapping between the slots of the schema(s) and the elements of the instantiation(s) and observing that no dissonance or mismatch arises. Sometimes a further inference must be drawn based on the structure of the resulting construction.

Each of the different types of statement is marked in proof discourse by a particular key word or phrase (marked in bold in the analysis below in Table 2).

3.2 Mental Operations in Interpreting Proof Statements

In both introducing and inference statements, interpretation involves making mappings and forming blends using structures from background knowledge and previous discourse, looking for dissonance or its absence, and drawing structural inferences. For any given statement, several of these elementary operations are required. Through introspection on the interpretation process, it is possible to identify the operations performed in interpreting any given statement and classify them. The types of elementary operations found thusfar through the examination of statements from proofs in analysis and algebra are listed in **Table 1**, below. The contents of the table are explained in detail in the context of an analysis of an actual mathematical proof in the next subsection.

3.3 Analysis of a Proof in Mathematical Analysis

Table 2 below contains the analysis of the first half of a proof in elementary point-set topology taken from Rudin (1976, p. 34). This book is a standard text used in introductory college courses in mathematical analysis. Point-set topology deals with the properties of sets of points

Table 1: Elementary Mental Operations for Proof Interpretation

CLASS	#	OPERATION
instantiating	1)	imagine entity exemplifying or satisfying a schematic slot
blend	2)	imagine entity involving blended combination of schemas/frames (usually embedding of extant, focused entity into a new frame)
focus	3)	focus on an element/substructure in an imagined schema, usually specified by a subschema
	4)	shift focus to structure present (focused on) in previous discourse – triggered by a) text, either explicitly or semiexplicitly (e.g., symbol in text is part of substructure of a previously given schema), or b) implicit cue
	5)	shift focus to a previous structure with new substructural detail (a, b variants as for #4)
observational blend	6)	(related to #4) blend in structure from previous discourse as substructure in current focused blend
	7)	blend a schema specified by text with contents in given context; 7': blend schema counterfactual to text with present contents
	8)	recall a previous blend and, using the same schema, add new substructure on instantiation side and map over to a new part of the schema
inspection	9)	note presence or absence of conflict/dissonance in immediate situation
	10)	note conflict with structure previously present in discourse
	11)	draw relevant schematic inference/notice relevant schematic implication
	12)	work sequentially through an analogy, verifying compatibility in mapping and drawing a resulting inference

in spaces that are generalizations of the coordinate spaces on which mathematical functions are defined.

In the table, the text of each step of the proof is given in the left-hand column, with the statement-type keyword in bold type. The second column lists the associated functional category of each step as given in Table 1 above. In the third and fourth columns the mental operations required to interpret the step (and accept its validity if it is an inference) are delineated. The third column lists the operation type under the classification in Table 2, and the fourth column provides a short description of how the operation is applied in the given situation, with all mathematical schemas marked by capitalization. The complete text of the proof as it appears in the book is given below. Note that there are no illustrations or diagrams associated with this proof, nor with any other in Rudin's text.

<u>Thm. 2.23</u>: *A set is open if and only if its complement is closed.*

> First, suppose E^c is closed. Choose $x \in E$. Then $x \notin E^c$, and x is not a limit point of E^c. Hence there exists a neighborhood N of x such that $E^c \cap N$ is empty, that is, $N \subset E$. Thus, x is an interior point of E, and E is open.

> Next, suppose E is open. Let x be a limit point of E^c. Then every neighborhood of x contains a point of E^c, so that x is not an interior point of E. Since E is open, this means that $x \in E^c$. It follows that E^c is closed.

In the remainder of this subsection, the interpretation of the first three steps of the proof is explained in detail, in order to provide a clear picture of the interpretation of both introduction and inference steps.

In the first substep of step 1, or step 1.1, we call to mind the schema for SET and combine it with the label 'E'. This can be analyzed using the blended spaces framework as in figure 2a. This blend is similar to the caused-motion example analyzed by Fauconnier & Turner (1994) in that the left-hand side contains a general schema and the right-hand side contains specific content items, in this case a single label and an attached entity standing schematically for the thing to be labeled. The "meaning" arrived at is that of a *specific instantiation* of the entity (called for by the right side) described by the schema (structure mapped from the left side).

In step 1.2, we blend the COMPLEMENT schema with two arguments (figure 2b). The first of these is the already-extant instantiation E, an entity compatible with one of the slots in the COMPLEMENT schema, and the second is 'E^c', another label. The result is that we have two instantiations held in mind, with a particular relationship between them specified by the schema.

In step 1.3, focus is shifted to this new entity, 'E^c', which has just been specified in relation to the previously focused E. This prepares the way for step 1.4 (not illustrated, but similar to 1.2), in which additional structure is given to the instantiation 'E^c' by mapping it into a compatible slot in the CLOSED schema.

In step 2.1 (see figure 3a), focus is shifted back to the SET instantiation E, triggered by the reference in the text. In step 2.2, focus is shifted a second time, to substructure of the SET schema that has just been recalled, and in step 2.3, a blend similar to that in step 1.2 is made associating the label 'x' with an instantiation of this substructure.

Thusfar, the operations have involved generating instantiations held in mind and bound to certain properties. The result is that an imagined configuration has been set up and in the next step (step 3) an observation is made on it. The text of the step states that a certain relationship that has not been specified in the preceding construction holds. The interpreter must look for evidence of the relationship in what is held in mind.

In step 3.1, focus is shifted back to the embedding of E and E^c into the COMPLEMENT schema. In step 3.2, a blend is constructed

Table 2: Analysis of a proof.

#	STEP	FUNC.	OP	DESCRIPTION
1.	First, **suppose** E^c is closed.	introduce	1	instantiate an entity (labeled 'E') in SET schema
			2	embed this in COMPLEMENT schema
			3	shift focus to another part of the schema (the complement, labeled E^c)
			2	embed this part in CLOSED schema
2.	**Choose** $x \in E$.	introduce	4a	return focus to 1a: E in SET schema
			3	focus on POINT subschema
			2	imagine instantiation ('x')
3.	**Then** $x \notin E^c$,	infer	5a	recall previous embedding of E into COMPLEMENT schema
			7	blend $\notin$ schema with arguments x,E^c
			9	see that no conflict/dissonance arises
			7'	counterfactually blend $\in$ schema with arguments x,E^c
			9	note that dissonance does arise
4.	**and** x is not a limit point of E^c.	infer	7'	blend LIMIT POINT schema with x, E^c
			4b	recall E^c CLOSED blend from 1d
			8	blend x (with contextual property LIMIT POINT) as subcontent into this
			11	draw inference $x \in E^c$ (schematic implication)
			10	note conflict with $x \notin E^c$ in earlier step
5.	**Hence** there exists a neighborhood N of x such that	infer	7	blend NEIGHBORHOOD schema with x
	$E^c \cap N$ is empty,		7	blend INTERSECTION schema with neighborhood and E^c
			4a'	embed into LIMIT POINT schema
			7'	counterfactually assume INTERSECTION nonempty
			11	draw inference that x is a LIMIT POINT
			10	note dissonance with previous step
6.	**that is**, $N \subset E$.	infer	4a'	from prior focus on N,E^c (as disjoint), recall COMPLEMENT schema
			3	use schema to shift focus to E
			2	blend SUBSET schema with args E,N
			9	note absence of dissonance
			7'	counterfactually assume N extends out (not SUBSET) of E
			11	draw inference (using COMPLEMENT schema) that $N \cap E^c$ nonempty
			10	note dissonance with step 3
7.	**Thus**, x is an interior point of E,	infer	7	blend INTERIOR POINT schema with x,E
			6	bring N (with context $\ni x$, $\subset E$) in from previous discourse
			9	note precondition satisfaction (absence of dissonance)
8.	**and** E is open.	infer	7	blend OPEN schema with x,E
			6	bring in x INTERIOR POINT from previous step
			9	note precondition satisfaction (dissonance absence)

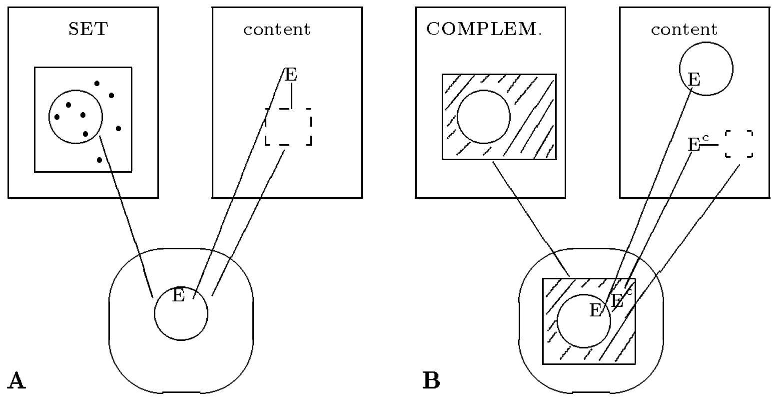

FIGURE 2 Blends in step 1 of proof. A: Step 1.1. B: Step 1.2.

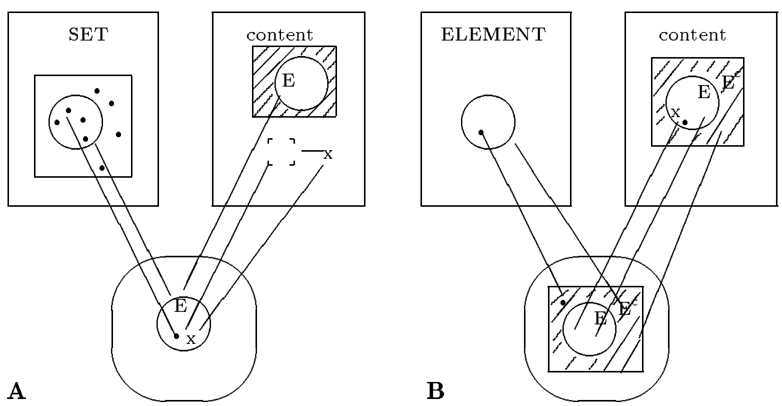

FIGURE 3 Blends in steps 2-3 of proof. A: Step 2.2. B: Step 3.4.

using the $\notin$ schema on the left-hand side and the instantiations x and E^c on the right-hand side. The difference between this type of blend (type 7) and blends of type 1 and 2 is that ALL content items on the right-hand side have prior bindings, hence the situation regarding potential mappings is more restricted. In this case, the restrictions do not generate a conflict with the specifications for the slots being mapped to in the $\notin$ schema, and this lack of dissonance is what is noted in step 3.3.

In most cases when similar slot-filler blends are made in natural language (as when mapping arguments into a construction), such an absence of dissonance suffices to accept the constructed meaning without further (or any) thought. However in mathematics, experience proves that dissonance may sometimes exist in such cases without being noticed (see following section), and one important check to eliminate this possibility is to construct the counterfactual blend and see that dissonance *does* exist in this case. Thus, a mathematician verifying the given proof would perhaps also carry out steps 3.4 and 3.5, attempting to construct a blend using a schema contrary to the specified one ($\in$ instead of $\notin$) and observing that conflict arises (from the structure of the COMPLEMENT schema) which prevents the mapping from being formed (see figure 3b).

4 Discussion

4.1 Parallels and Differences Between Mathematics and Language

The preceding analysis has shown that interpretation of natural language and mathematical proofs exhibit several parallels. In both cases the structures manipulated are ultimately grounded in experience, we can identify conceptual mapping and blending operations in the construction of meaning, and the retrieval of structures from discourse context and background knowledge is often required. An important difference, however, is that conscious awareness of underlying conceptual operations is greater in mathematics than language. One reason for this is that the schemas employed in blends in the former case are conceptually novel structures that have been consciously constructed by mathematicians, rather than subconsciously acquired from regularities in linguistic and extralinguistic experience. Another reason is that whereas in language it is generally only the results of blending that are attended to, in mathematics conscious attention must be paid to whether or not constructed blends actually *work* – that is, whether or not dissonance arises as a result of the mapping.

4.2 Understanding Trends in the Development of Mathematics

Owing to the parallels in grounding and manipulation, interpretation in informal mathematics is subject to some of the same difficulties inherent in language. In particular, there are two potential sources of miscommunication leading to disagreement over the validity of theorems and proofs, depending on whether the problem lies with the schematic entities or their manipulation.

The problem with schemas lies in their specification via natural language definitions. If the wording of a given definition is not sufficiently precise, there may be more than one conceptual schema that is compatible with it. In this case different mathematicians may be using the same terminology to refer to different schemas with different interactive properties, leading to confusion and disagreement over which relationships hold.

The second kind of difficulty arises in the manipulation of schemas that occurs in proofs. As shown in the preceding section, instantiations must be mapped into slots and their eligibility for the slots verified. Often several factors (required properties) bear on such eligibility, and as the situation grows more complex, it becomes more difficult to hold everything clearly in mind without making any oversights. The more complex the schemas and their slot requirements, the more difficult and error-prone these kinds of operations will be.

During the 18th and particularly the 19th centuries, awareness of these difficulties led to modifications of both mathematical concepts and the complexity of the inference steps used in their manipulation (Bell, 1945[4]). Mathematical concepts were defined more precisely and by appeal to simpler schemas from ordinary experience.

These simplifications to concepts in turn affected reasoning in proofs, it being easier to handle concepts and examine the interactions of their properties when they are simpler. And when the concepts were complex, mathematicians were more careful to split reasoning into smaller steps with less needing to be verified in each step. This was the basis of the trend towards increasing rigor in mathematics.

The culmination of these trends of increasing precision and rigor came at the beginning of the 20th century, with the development of the formal mathematical program by Hilbert (1899/1959) and its most thorough implementation to date, within a symbolic logic framework, by Whitehead & Russell (1910/1925). Although the immediate stimulus for the development of Hilbert's program was the apparent existence of conflicts between the different types of non-Euclidean geometry that had been discovered (see Bell, 1945, chapter 15 (pp. 332–5), chapter 23), it would not have occurred without the stage having been set by the trends.

5 Conclusions and Future Directions

In this paper, a description of the grounding of mathematical concepts in everyday experience elaborating on ideas of Lakoff and Johnson has been put forth, and some proposals have been made as to how these conceptual structures interact in the interpretation of mathematical proofs, based on Fauconnier & Turner's blended spaces framework. The analysis resulting from this proposal highlights two important observations. First, the cognitive operations involved in interpreting proofs do

[4]This book, though inaccurate in some technical details, provides an excellent survey. See in particular chapters 13 and 23.

not seem qualitatively different from those involved in ordinary natural language, and, second, the structure of these operations is generally more consciously accessible in mathematics than in language.

This suggests that one direction to extend the present work would be to utilize the analysis of mathematical discourse as a *window* on processes in natural language interpretation. For example, the fine detail of patterns of reference to discourse context is accessible to introspection in mathematical proof, and it was even possible to classify focus-shifts and blending operations on this basis. Another area which might potentially benefit is the understanding of acceptability constraints for conceptual mappings (see Turner, 1991). Such constraints are highly accessible to introspection in mathematical reasoning, because it is often the question of whether a given mapping is acceptable or not that is focused on in verifying a proof step.

This analysis might also serve as a window on human reasoning in general. We suggest that interpretive reasoning in mathematics proceeds by the coordinated interaction of schematic structures – by means of structure mapping, focus-shifting, and dissonance searching. To the extent that reasoning in other domains such as science or law is similar to that in mathematics, this would provide a structured way of conceptualizing the underlying cognitive processes in those domains.

References

Bell, E.T. 1945. *The Development of Mathematics*. New York: McGraw-Hill.

Bundy, A. 1983. *Computer Modelling of Mathematical Reasoning*. New York: Academic Press.

Farmer, W.M., J.D. Guttman, and F.J. Thayer. 1995. Contexts in Mathematical Reasoning and Computation. *J. Symbolic Computation* 19:201–16.

Fauconnier, G., and M. Turner. 1994. Conceptual Projection and Middle Spaces. Technical Report 9401. Department of Cognitive Science, University of California, San Diego. available from `http://cogsci.ucsd.edu`.

Hardy, G.H. 1928. Mathematical Proof. *Mind* 38:1–25.

Heath, T.L. (ed.). 1956. *The Thirteen Books of Euclid's Elements*. New York: Dover Publications.

Hilbert, D. 1959. *The Foundations of Geometry*. La Salle, Ill.: Open Court Pub. Co. originally published Grundlagen der Geometrie, 1899.

Johnson, M. 1987. *The Body in the Mind*. Chicago: University of Chicago Press.

Kitcher, P. 1984. *The Nature of Mathematical Knowledge*. Oxford: Oxford University Press.

Lakatos, I. 1976. *Proofs and Refutations*. Cambridge: Cambridge University Press.

Lakoff, G. 1987. *Women, Fire, and Dangerous Things.* Chicago: University of Chicago Press.

Peano, G. 1889. *Principii di Geometrica Logicamente Eposti.* Torino: Fratelli Bocca.

Rudin, W. 1976. *Principles of Mathematical Analysis, 3rd Edition.* New York: McGraw-Hill.

Turner, M. 1991. *Reading Minds.* Princeton, N.J.: Princeton University Press.

Whitehead, A.N., and B. Russell. 1925. *Principia Mathematica, 2nd Edition.* Cambridge: Cambridge University Press. 1st Edition published 1910.